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ADAPTIVE STATE VECTOR CONTROL

SECTION 4

THE THEORY OF TIME-OPTIMAL CONTROL OF LINEAR SYSTEMS

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FOREWORD

This document is the fourth of nine sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Cpmpany for the National Aeronautics and Space Administration under Contract NASr-27. The report is issued in the following nine sections to facilitate updating as progress warrants:

1529-TR1	An Introduction to Self-Evaluating State Vector Control of Linear Systems
1529-TR2	Modes of Control
1529-TR3	Measurement of the State Vector
1529-TR4	The Theory of Time-Optimal Control of Linear Systems
1529-TR5	Computational Solution of Optimal Control Problems
1529-TR6	The Theory of Average Power Optimal Control of Linear Systems
1529-TR7	Approximations to State Vector Control
1529-TR8	A Logical Net Mechanization for Time-Optimal Regulation
1529-TR9	Adaptive Controllers Derived by Stability Considerations

Section 1 (1529-TR1) provides the motivation for the research effort, defines the problems, indicates the status, and presents computer results to show what can feasibly be accomplished at the present time. The computational technique which is used to obtain trajectories satisfying the Maximum Principle is described.

In section 2 it is shown that, given the vector differential equation $\dot{x} = Ax + bu$ (where x is an n vector, A an nxn constant matrix, b an nxl constant matrix, and u a scalar), if any m components of the state vector can be brought to zero in finite time, a constructive method exists for reformulating the problem to that of bringing a single component of the state vector to zero.

Section 3 shows that if a finite number of flexure modes are sufficient to represent the aeroelastic distortions of a flexible vehicle, the state vector of an aerial vehicle can be measured with commonly available instrumentation. An approximate method for removing the flexure effects for the rigid body motion is also presented.

Theory of time-optimal control of linear plants that can be represented by linear differential or linear recurrence equations is presented in section 4.

Another state vector control theory is presented in section 5. It is shown that it can be specialized to yield a method for computing trajectories for the special time-optimal regulation problem where all components of the state vector are driven to the origin. Alternatively, it permits computation of trajectories for driving all components of the state vector to the origin under an amplitude restriction on one of the state vector components if the forcing function is not restricted.

State vector regulation with an average power constraint on the forcing function is considered in section 6. Besides having merit in its own right, the method is shown to yield a good approximation to several kinds of time-optimal regulators.

Most aerial vehicles do not require the performance that can be achieved with state vector control. Section 7 presents some approximation methods so state vector control theory can be used to obtain adequate performance without inordinate complexity.

In section 8 a method is presented that provides a feasible mechanization for time-optimal regulators.

The self-evaluating and adaptive control problems are considered in section 9.

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THE THEORY OF TIME-OPTIMAL CONTROL OF LINEAR SYSTEMS

By C. A. Harvey and E. B. Lee

SUMMARY

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An open-loop time-optimal regulator problem is considered for plants that can be represented by linear differential or linear recurrence equations. The problem is treated in sufficient generality to encompass both single- and multiple-component regulator problems. Uniqueness theorems are proved in order to derive necessary and sufficient conditions that an optimal control must satisfy.

In the discussion of differential equation systems, the single component control problem is considered first. It is shown that this problem gives rise to a target set which is compact and convex. This set is contained in a p-dimensional subspace of the phase space and has an interior relative to this subspace. The dimension p depends on the particular problem. If p is greater than two, it is not possible, in general, to obtain explicit formulas for the boundary of the target set.

While the control problem considered here is open loop, it is clear that open- and closed-loop trajectories coincide when the time-optimal criterion is used. Thus the uniqueness theorems provide the conditions required for open- or closed-loop control. The main result is that the control must satisfy Pontryagin's maximum principle with the associated transversality condition. For second-order systems explicit formulas for closed-loop control can be obtained. A second-order example is presented.

For plants represented by linear autonomous recurrence equations, the control problem analogous to that considered for plants represented by linear autonomous differential equations is considered after the elementary theory of linear recurrence equations is presented. The results for recurrence equation systems are analogous to those obtained for differential equation systems.

INTRODUCTION

The all-component time-optimal control problem has been studied by various authors. For linear systems the resulting theory is quite advanced. The single- and multiple-component time-optimal control problems have not received such wide attention. These problems provide the motivation for the development of the time-optimal control theory given below.

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For systems represented by linear differential equations it is shown in section 2 (MH MPG Report 1529-TR2) that multiple-component time-optimal regulation can be accomplished by a single-component time-optimal regulator. The single-component time-optimal regulator problem is reformulated below so that it is in the form of the general problem statement given subsequently. The all-component time-optimal regulator problem may also be stated in terms of the general problem statement.

Background results are given for this general problem. Then theorems concerning the uniqueness of optimal and extremal controls are proved. The methods of proof are based on geometrical considerations.

Systems represented by linear recurrence equations are then considered. For such systems a time-optimal control theory is developed which is analogous to the theory developed for systems represented by linear differential equations. Before the control problem is considered, the elementary theory of recurrence equations is developed for completeness of the presentation.

SYSTEMS REPRESENTED BY LINEAR DIFFERENTIAL EQUATIONS

In section 2 it is shown that any multiple-component time-optimal regulator problem for a linear system with a single control variable is equivalent to a single-component time-optimal regulator problem. For this reason, only the single-component regulator problem need be considered.

For the general problem statement given below, which encompasses the single-component time-optimal regulator problem, it is shown that if the hypotheses of theorem 2 are satisfied then Pontryagin's maximum principle and associated transversality condition form necessary and sufficient conditions that are to be satisfied by an optimal control. Thus, the character of optimal controls is given and a set of transcendental equations may be obtained. The solution of these equations defines the optimal control.

Single-Component Control Problem

This problem may be stated in the following manner: Let c(t) represent the component to be controlled and u(t) represent the control variable where c(t) and u(t) are scalars. Suppose that u(t) is constrained to satisfy $|u(t)| \leq 1$ and that c(t) and u(t) are related by the linear differential equation

$$\sum_{i=0}^{n} a_i c^{(i)}(t) = \sum_{j=0}^{m} b_j u^{(j)}(t) \tag{1}$$

where $m \leq n$, a_i, b_j constants, $a_n = b_m = 1$. Then the single-component time-optimal regulator problem is to find u(t) for $t > 0$ satisfying $|u(t)| \leq 1$

such that $c(t) = 0$ for $t > T$ where T is a minimum for any given set of initial conditions $c^{(i)}(0) = c_o^{(i)}$, $u^{(j)}(0) = u_o^{(j)}$, $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$.

It should be noted that $c(t) = 0$ for $t > T$ if, and only if, $c^{(i)}(T^+) = 0$, $i = 0, 1, \ldots, n-1$ and $u(t)$ satisfies $|u(t)| \leq 1$ and

$$\sum_{j=0}^{m} b_j u^{(j)}(t) = 0 \tag{2}$$

for $t > T$. Let $\overline{G}$ be the set of points in Euclidean m-space with coordinates $g_1, g_2, \ldots, g_m$ such that the solution of (2) with $u^{(j)}(T^+) = g_{j+1}$, $j = 0, 1, \ldots, m-1$ satisfies $|u(t)| \leq 1$ for $t > T$. Evidently $\overline{G}$ is compact, convex, and nonempty.

Now a transformation will be introduced to transform equation (1) to the vector form

$$\dot{x} = Ax + Bu$$

where A and B are nxn and nxl constant matrices, respectively, and the problem will be reformulated in terms of this system. Let α_{ij}, $i = 0, 1, \ldots, n$; $j = 0, 1, \ldots, m-1$ be defined by the equations

$$\sum_{i=o}^{n} \alpha_{ij} a_i = b_j, \quad j = 0, 1, \ldots, m-1$$

$$\alpha_{n-1,m-1} = 1, \quad \alpha_{i,m-1} = 0, \quad i < n-1 \tag{3}$$

$$\alpha_{i+1,j+1} = \alpha_{ij}, \quad i = 0, 1, \ldots, n-1, \quad j = 0, 1, \ldots, m-2$$

Then α_{ij} represents the jump in $c^{(i)}(t)$ caused by a unit jump discontinuity in $u^{(j)}(t)$; i. e.,

$$\alpha_{ij} = c^{(i)}(t^+) - c^{(i)}(t^-) \tag{4}$$

when

$$u^{(k)}(t^+) - u^{(k)}(t^-) = \delta_{kj}$$

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where

$$\delta_{kj} = 1 \text{ if } k = j \text{ and } \delta_{kj} = 0 \text{ if } k \neq j$$

Introducing

$$x_{i+1}(t) = c^{(i)}(t) - \sum_{j=o}^{m-1} \alpha_{ij} u^{(j)}(t), \; i = 0, 1, \ldots, n-1 \tag{5}$$

equation (1) becomes

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) + \alpha_{i,o} u(t), \; i = 1, 2, \ldots, n-1 \\ \dot{x}_n(t) &= -\sum_{i=o}^{n-1} a_i x_{i+1}(t) + \alpha_{n,o} u(t) \end{aligned} \tag{6}$$

Initial conditions transform according to equation (5).

The conditions $c^{(i)}(T^+) = 0, \quad i = 0, \quad 1, \ldots, n-1$ give

$$x_{i+1}(T^+) = -\sum_{j=o}^{m-1} \alpha_{ij} u^{(j)}(T^+), \quad i = o, \quad 1, \ldots, n-1 \tag{7}$$

from (5), but each $x_i(t)$ is continuous so that $x_i(T^+) = x_i(T) = x_i(T^-)$. Equation (7) defines a linear mapping, L, of an m-dimensional vector space into an n-dimensional vector space. Let G be the image of $\overline{G}$ under this map, i. e., $G = L\overline{G}$. Then the condition $c(t) = 0$ with $u(t)$ satisfying $|u(t)| \leq 1$ for $t > T$ is equivalent to: $x(T)$ is in G and $u(t)$ is the solution of equation (2) for $t > T$ with $u^{(j)}(T^+)$ satisfying equation (7). Thus the single-component time-optimal regulator problem is reformulated as follows: Given the system described by equation (6), for any set of initial conditions it is desired to:

a. Choose $u(t)$, satisfying $|u(t)| \leq 1$ for $0 \leq t \leq T$, such that the corresponding $x(t)$ reaches G at $t = T$ where T has the property that for any $u(t)$ satisfying $|u(t)| \leq 1$ for $0 \leq t \leq T$ the corresponding $x(t)$ is not in G for $t < T$.

b. For $t > T$ let $u(t)$ be the solution of equation (2) with $u^{(j)}(T^+)$ satisfying equation (7).

If the solution for (a) is found, there is a solution for (b) because of the definition of G. This solution of (b) is uniquely determined by x(T) and is given by $u(t) = 0$ if $m = 0$ and $u(t) = -x_{n+1-m}(t)$ if $m > 0$ where $x_{n+1-m}(t)$ is determined by

$$\dot{x}_i = x_{i+1} - \alpha_{i,o}\, x_{n+1-m}, \quad i = 1, 2, \ldots, n-1$$

$$\dot{x}_n = -\sum_{i=o}^{n-1} a_i x_{i+1} - \alpha_{n,o}\, x_{n+1-m} \tag{8}$$

with initial condition x(T) which is given by (a).

Remark: Although $\overline{G}$ and G are well defined, it may be impossible to obtain explicit formulas for their boundaries. If the dimension of G is less than three, explicit formulas for the boundary of G can be obtained.

General Problem Statement and Background Results

Consider the autonomous linear control problem **P**:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k; \quad i = 1, 2 \ldots n \tag{9}$$

the nonempty restraint set $\Omega = \{(u_1, \ldots, u_k): |u_k| \leq 1, k = 1, 2, \ldots, m\}$

the initial point $x_o = (x_{1o}, x_{2o}, \ldots, x_{no}) \in R^n$,

the constant, compact, convex target set is $G \in R^n$,

the cost functional of control $C(u) = t_1$.

The problem of optimum control is to select $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]$ from Ω for each t, $0 \leq t \leq t_1$, such that $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ moves from $x(0) = x_o$ to an intersection with the target G in minimum time $t_1 = C(u)$. A control u is called optimal in case $C(u) \leq C(\hat{u})$ for any admissible control $\hat{u} \in \Delta$, where Δ is the class of all measurable controls $u(t) \in \Omega$ on various time intervals $0 \leq t \leq t'$ which "steer" x(t) from x_o to G.

Pontryagin (ref. 1), et al. have shown that a time-optimal control is necessarily extremal; that is

1. There exists an absolutely continuous vector that is nowhere zero,

   $\eta(t) = [\eta_1(t), \eta_2(t), \ldots, \eta_n(t)]$

   on $t_o \leq t \leq t_1$ such that $x(t)$, $\eta(t)$, $u(t)$

satisfy the differential equation system

$$\dot{x}_i = \frac{\partial H}{\partial \eta_i}$$

$$\dot{\eta}_i = -\frac{\partial H}{\partial x_i}\,; \quad i = 1, 2 \ldots n \tag{10}$$

2. $H\left[\eta(t), x(t), u(t)\right] = M\left[\eta(t), x(t)\right]$ for almost all t on $t_o \leq t \leq t_1$,

where

$$H = \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} a_{ij}\, x_j + \sum_{i=1}^{n} \eta_i \sum_{k=1}^{m} b_{ik}\, u_k \tag{11}$$

and

$$M(\eta, x) = \operatorname*{Max}_{u \in \Omega} H(\eta, x, u) \tag{12}$$

3. $M\left[\eta(t_o), x(t_o)\right] = M\left[\eta(t), x(t)\right] \geq 0$ for all $t_o \leq t \leq t_1$ and any $\eta(t)$, $x(t)$, satisfying conditions 1 and 2. (13)

For certain of the results given here, it is necessary to assume that system (9) satisfies the following condition:

Definition: System (9) is called <u>normal</u> if the vectors Bw, ABw, . . . , A^{n-1} Bw are linearly independent, where w is a vector having the direction of an edge of the polyhedron Ω (or along Ω itself if Ω is just a line segment).

If system (9) is normal, there is only one point $u \in \Omega$ at which

$$F(u) = \sum_{i=1}^{n} \eta_i \sum_{k=1}^{m} b_{ik}\, u_k$$

assumes its maximum for almost all t (see for example Pontryagin, (ref. 6). Thus, if $\eta(t)$ is known, the control u(t) is uniquely determined and is given by

$$u_k(t) = \operatorname{sgn}\left[\sum_{i=1}^{n} \eta_i(t)\, b_{ik}\right] \tag{14}$$

of in vector notation $u(t) = \operatorname{sgn}\left[\eta(t) \cdot B\right]$ for the normal system $(9)^*$. It is desired to establish results concerning the uniqueness of the controls of the form (14) for the problem P.

Rozonoer (ref. 2) has shown that the vector $\eta(t)$ for the optimum control progrm P must also satisfy a transversality condition; that is, when $x(T)\epsilon\partial G$, the vector $\eta(T)$ must be normal to a supporting hyperplane of the set G at the point $x(T)$ and directed into the halfspace containing G. (The necessity of this condition for optimal control is obvious from lemma 5.)

Uniqueness Theorems

As is well known (ref. 3) the solution to equation (9) can be written as the integral equation

$$x(t) = e^{At} x_o + \int_o^t e^{A(t-s)} B\, u(s)\, ds, \quad x_o = x(0)$$

and the solution to equation (10) as

$$\eta(t) = e^{-A't} \eta_o, \quad \eta_o = \eta(0)$$

where $e^{At} = I + tA + (t^2A^2/2!) + \ldots$, and ' indicates transpose.

For each $t \geq 0$, let

$$K(t, x_o) = \left[x : x = e^{At} x_o + \int_o^t e^{A(t-s)} B\, u(s)\, ds, \ u(s) \text{ allowable} \right]$$

It is desired to establish certain properties of $K(t,x_o)$ for the normal systems which will be of aid in proving the existence and uniqueness theorems. Evidently (ref. 4) $K(t,x_o)$ is compact and convex, and the boundary of $K(t,x_o)$ denoted by $\partial K(t,x_o)$ is given by

$$\partial K(t, x_o) = \left[x : x = e^{At} x_o + \int_o^t e^{A(t-s)} B \operatorname{sgn} (\eta \cdot B)\, ds; \ \|\eta_o\| = 1 \right]$$

Lemma 1. - For each $\eta(t)$ with $\|\eta_o\| = 1$

$$\eta(t) \cdot (e^{At}) \left\{ x_o + \int_o^t e^{-As} B \operatorname{sgn} \left[\eta(s) \cdot B \right] ds \right\} \geq \eta(t) \cdot \zeta$$

for all $\zeta \epsilon K(t, x_o)$, and equality holds only if

$$\zeta = e^{At} \left\{ x_o + \int_o^t e^{-As} B \operatorname{sgn} \left[\eta(s) \cdot B \right] ds \right\} \text{ for the normal system (9)}$$

*If $\eta(t)$ is any nontrivial solution of equation (10), then $\eta(t) \cdot B$ has a finite number of zeros on $0 \leq t \leq t_1$.

Proof:

$$\eta(t) \cdot e^{At}\left\{x_o + \int_o^t e^{-As} \operatorname{sgn}\left[\eta(s) \cdot B\right] ds\right\} - \eta(t) \cdot \zeta$$

$$= \eta_o \cdot \left[\int_o^t e^{-As}B \operatorname{sgn}(\eta_o \cdot e^{-As}B)\, ds\right] - \eta_o \cdot \left[\int_o^t e^{-As}B\, u(s)\, ds\right]$$

$$= \int_o^t \left[\eta_o \cdot e^{-As}B \operatorname{sgn}(\eta_o \cdot e^{-As}B) - \eta_o \cdot e^{-As}B\, u(s)\right] ds$$

$$= \int_o^t \sum_{i=1}^m \left[|f_i(s)| - f_i(s)u_i(s)\right] ds = \sum_{i=1}^m \int_o^t \left[|f_i(s)| - f_i(s)u_i(s)\right] ds$$

where $f_i(s)$ = i^{th} component of the row vector $\eta_o \cdot e^{-As}B$. But

$$|f_i(s)| - f_i(s)u_i(s) \geq |f_i(s)| - |f_i(s)u_i(s)| \geq |f_i(s)| - |f_i(s)| = 0 \text{ for } i = 1, 2, \ldots m.$$

Therefore,

$$\sum_{i=1}^m \int_o^t \left[|f_i(s)| - f_i(s)u_i(s)\right] ds \geq 0$$

If equality holds, then for $i = 1, 2, \ldots m$

$$\int_o^t \left[|f_i(s)| - f_i(s)u_i(s)\right] ds = 0$$

which implies $|f_i(s)| = f_i(s)u_i(s)$ a. e., on $(0, t)$ since $|f_i(s)| - f_i(s)u_i(s) \geq 0$. Since the system is normal, $|f_i(s)| = f_i(s)u_i(s)$ a. e., on $(0, t)$ implies $u_i(s) = \operatorname{sgn}[f_i(s)]$ a. e. on $(0, t)$ which gives

$$\zeta = e^{At} x_o + \int_o^t e^{A(t-s)}B \operatorname{sgn}\left[\eta(s) \cdot B\right] ds$$

Lemma 2. - The boundary of $K(t, x_o)$ contains no line segments for the normal system (9).

Proof: Suppose the boundary of $K(t, x_o)$ contains a line segment L joining the points x^1 and x^2. Choose a supporting plane, π, to $K(t, x_o)$ at x^1 containing L and take η to be the unit vector normal to π directed into the halfspace not containing $K(t, x_o)$. Then $\eta \cdot x^1 = \eta \cdot x^2$ and $x^1 \neq x^2$. Hence there exists $x^3 \epsilon K(t, x_o)$

such that $\eta \cdot x^3 > \eta \cdot x^1$; i. e., x^3 lies on the opposite side of π from the set $K(t, x_o)$, which is a contradiction because π is a supporting plane to $K(t, x_o)$.

Lemma 3. - Consider the sets $K(t, x_o)$, $0 \leq t \leq t^*$. Corresponding to each $\epsilon > 0$ there is a $\delta > 0$ such that $d[p, K(t, x_o)] < \epsilon$(d as defined below) for each $p \epsilon K(t^*, x_o)$ and all $t^* - \delta < t < t^*$.

Proof: Let $p \epsilon K(t^*, x_o)$ and $q \epsilon K(t, x_o)$, $0 < t < t^*$. Then there exists admissible up(s) and uq(s) defined on the intervals $(0, t^*)$ and $(0, t)$, respectively, such that

$$p = e^{At^*}\left[x_o + \int_o^{t^*} e^{-As} B\, u_p(s)\, ds\right]$$

and

$$q = e^{At}\left[x_o + \int_o^{t} e^{-As} B\, u_q(s)\, ds\right]$$

Let

$$q_p = e^{At}\left[x_o + \int_o^{t} e^{-As} B\, u_p(s)\, ds\right]$$

Then

$$q_p\, \epsilon K(t, x_o) \text{ and } d\left[p, K(t, x_o)\right] = \min_{q \epsilon K(t, x_o)} \|p - q\| \leq \|p - q_p\|$$

But

$$\|p - q_p\| = \left\|\left[e^{A(t^*-t)} - I\right]\left[e^{At}x_o + \int_o^t e^{A(t-s)}Bu_p(s)\, ds\right] + \int_t^{t^*} e^{A(t^*-s)}Bu_p(s)ds\right\|$$

Thus,

$$d\left[p, K(t, x_o)\right] \leq \left\|e^{A(t^*-t)} - I\right\|\left[\left\|e^{At}x_o\right\| + \left\|\int_o^t e^{A(t-s)}Bu_p(s)ds\right\|\right] + \left\|\int_t^{t^*} e^{A(t^*-s)}Bu_p(s)ds\right\|$$

Let $C_1 = \max_{t \epsilon [0, t^*]} \|e^{At}x_o\|$

which exists since $\|e^{At}x_o\|$ is continuous in $[0, t^*]$.

There is a C_2 such that

$$\left\| \int_0^t e^{A(t-s)} Bu_p(s)\, ds \right\| \leq C_2$$

for all $t \in [0, t^*]$ and all admissible $u_p(s)$ defined on $[0, t]$ since

$$\int_0^t e^{A(t-s)} Bu_p\, ds \in K(t, 0) \subset K(t^*, 0)$$

which is compact (ref. 5).

Also there is a C_3 such that $\| e^{A(t^*-s)} Bu_p(s) \| \leq C_3$ for all $s \in [0, t^*]$ and $u_p(s)$ admissible.

Hence,

$$d[p, K(t, x_o)] \leq \| e^{A(t^*-t)} - I \| (C_1 + C_2) + (t^* - t) C_3$$

for each $p \in K(t^*, x_o)$.

The right side of this inequality is continuous in t and takes the value zero when $t = t^*$, thereby establishing the lemma. Thus, the sets $K(t, x^o)$, $0 \leq t \leq t^*$ satisfy the hypotheses of the following lemma of ref. 5, page 6:

"Lemma 3. - Consider sets $A(t)$ of R^n, $t \leq t^*$, with the following properties:

a. Each $A(t)$ is convex

b. Corresponding to each $\epsilon > 0$ there is a $\delta > 0$ such that $d[p, A(t)] < \epsilon$ for each $p \in A(t^*)$ and all $t^* - \delta < t < t^*$. Then, if q is in the interior of $A(t^*)$, there is a $t_1 < t^*$ such that q is an interior point of $A(t_1)$."

Theorem 1. The uniqueness of the optimal controls for the problem P is established by the following theorem: Let there be given a compact, convex set G of R^n. For the normal system (9) suppose $u^1(t)$ and $u^2(t)$ are such that the corresponding trajectories initiating at $x(0) = x_o$ intersect G time-optimally at $t = t^*$. Then

$$u^1(t) = u^2(t) \text{ a.e. on } (0, t^*)$$

Proof: Let $x^i(t)$ denote the solutions corresponding to $u^i(t)$ for $i = 1, 2$. Then $x^1(t^*)$ and $x^2(t^*)$ are on the boundary of $K(t^*, x^o)$ and also on the boundary of G. Assume $x^1(t^*) \neq x^2(t^*)$. Then the line segment L which joins $x^1(t^*)$ to $x^2(t^*)$, is also in G and in $K(t^*, x^o)$. If any point of L were an interior point to $K(t^*, x^o)$, it would be interior to $K(t, x^o)$ for some $t < t^*$ which gives a point of G attainable for $t < t^*$ which contradicts the hypothesis. Therefore, L must lie on the boundary of $K(t^*, x^o)$ which contradicts lemma 2. Thus $x^1(t^*) = x^2(t^*)$. Hence by theorem 4 of Pontryagin's paper (ref. 6), $u^1(t) = u^2(t)$ a. e. on $(0, t^*)$.

Further properties of $K(t, x_o)$ are given in the following lemmas.

Lemma 4. - Let $\omega \in K(t, x_o)$.

If

$$\omega = e^{At}\left[x_o + \int_o^t e^{-As}B \operatorname{sgn}(\eta_o^{\,i} \cdot e^{-As}B)\, ds\right]$$

for i = 1, 2, then

$$\operatorname{sgn}(\eta_o^{\,1} \cdot e^{-As}B) = \operatorname{sgn}(\eta_o^{\,2} \cdot e^{-As}B), \quad 0 < s < t$$

a. e. for the normal system (9).

Proof:

$$e^{At}\left[x_o + \int_o^t e^{-As}B \operatorname{sgn}(\eta_o^{\,1} \cdot e^{-As}B)\, ds\right] - e^{At}\left[x_o + \int_o^t e^{-As}B \operatorname{sgn}(\eta_o^{\,2} \cdot e^{-As}B)\, ds\right] = 0.$$

Then

$$\eta_o^{\,1} \cdot \int_o^t e^{-As}B \operatorname{sgn}(\eta_o^{\,1} \cdot e^{-As}B)\, ds - \eta_o^{\,1} \cdot \int_o^t e^{-As}B \operatorname{sgn}(\eta_o^{\,2} \cdot e^{-As}B)\, ds = 0$$

or

$$\int_o^t \left| \eta_o^{\,1} \cdot e^{-As}B \right| ds = \int_o^t \eta_o^{\,1} \cdot e^{-As}B \operatorname{sgn}(\eta_o^{\,2} \cdot e^{-As}B)\, ds\,,$$

This implies that

$$\operatorname{sgn}(\eta_o^{1} \cdot e^{-As}B) = \operatorname{sgn}(\eta_o^{2} \cdot e^{-As}B)$$

a. e. for the normal system (9).

Lemma 5. - For each point

$$\omega(\eta_o) = e^{At}\left[x_o + \int_o^t e^{-As}B \ \operatorname{sgn}(\eta_o \cdot e^{-As}B)\, ds\right]$$

on the boundary of $K(t, x_o)$ where $\| \eta_o \| = 1$, $\eta(t) = e^{-A't} \eta_o$ is an exterior normal at $\omega(\eta_o)$.

Proof: $\eta(t) \cdot [\omega(\eta_o) - x] = 0$ is the equation of a hyperplane, π, passing through $\omega(\eta_o)$, which is orthogonal to $\eta(t)$. From lemma 1, $\eta(t) \cdot [\omega(\eta_o) - \omega] \geq 0$ for all ω in $K(t, x_o)$; therefore, π supports $K(t, x_o)$. From the sense of the inequality it is clear that $\eta(t)$ has the direction exterior to $K(t, x_o)$ at $\omega(\eta_o)$.

The uniqueness of the extremal control for the problem P is provided by Theorem 2.

Theorem 2. - Consider the control problem P. Let $u \epsilon \Delta$ be any control function satisfying equation (14) where $\eta(t)$ satisfies equation (10) and the transversality condition. If $G \cap [K(t, x_o) - \partial K(t, x_o)] \neq \phi$ for all $t > t^*$ where

$$t^* = \inf_{u \epsilon \Delta} \left[C(u)\right]$$

then u(t) is the unique optimal control for the normal system (9).

Proof: Consider a common supporting hyperplane π^* to the sets G and $K(t^*, x_o)$. Let $\eta(t^*)$ be a vector normal to π^* directed into the halfspace containing G. By theorem 1 and lemmas 4 and 5, $\eta(t^*)$ uniquely determines the optimum control $u^*(t)$. It is desired that $u^*(t)$ is the only extremal control which steers x_o to G and satisfies the transversality condition. (It is only necessary to consider boundary points of G in looking for optimal controls.) At a point x(t) on ∂G, $\eta(t)$ is normal to a supporting hyperplane of G at x(t) and directed into the halfspace containing G. (This is the transversality condition.) For x(t) on $\partial K(t, x_o)$, $\eta(t)$ is normal to a supporting hyperplane of $K(t, x_o)$ at x(t) and is directed into the halfspace not containing $K(t, x_o)$, according to lemma 5. Consider any time $t_1 > t^*$ at which $K(t_1, x_o)$ and G again have a common supporting hyperplane π^1. $\eta(t_1)$ must be normal to π^1 and be directed out of the halfspace containing $K(t_1, x_o)$ into the halfspace containing G. This is impossible since the two convex sets G and $K(t_1, x_o)$ must be on the same side of π^1 if $G \cap [K(t_1, x_o) - \partial K(t_1, x_o)] \neq \phi$. Therefore, $t_1 \leq t^*$, but t^* is the first t for which x(t) is in G, so $t_1 = t^*$.

Remark: $G \cap [K(t_i, x_o) - \partial K(t_i, x_o)] \neq \phi$, $t_i > t^*$ for the problem P if the vector $\dot{x}$ is directed into the set G for every point on the boundary of G for some choice of $u \epsilon \Omega$.

Synthesis of the Optimum Control

It has been demonstrated by Lee and Markus (ref. 7) (also by Pontryagin, ref. 8, for the linear problem) that if the set Δ is nonempty*, an optimum control will exist for the problem P. Obviously, the set Δ is nonempty for any x_o in R^n if G contains the origin in R^n and if the matrix A of the normal system (9) is stable (i. e., if all of the roots of $|A - \lambda I| = 0$ have negative real parts). This follows from theorem 7 of ref. 6 or the corollary on page 6 of ref. 7. Since anything which can be done with allowable control can also be done with optimal control, the synthesis procedure to be outlined will provide the domain of controllability, that is, the set of points (initial conditions x_o) from which the set G can be reached using allowable control. This will then completely answer the question of existence.

Theorem 2 is of interest in solving the synthesis problem. The procedure used is the same as that suggested by LaSalle in ref. 8. It is assumed that system (9) and set G satisfy the hypothesis of theorem 2. Then system (9) can be started at points on the boundary of G, and system (10) and equation (14) can be used to determine the control function u(t), observing the solution as t decreases (replace t by -t) for the various allowable initial conditions on $\eta(t)$ as determined by the transversality condition with $\| \eta_o \| = 1$. The control function as determined is then the optimal control for all points x(t) which can be attained in this manner. This is then a constructive procedure for determining the switching surface, where the components of u(t) change signs as functions of $x = (x_1, x_2 \ldots x_n)$; that is, u as $u(x_1, x_2 \ldots x_n)$ for the control problem P.

As an example, consider the second-order problem:

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= -x_2 + u \end{aligned} \tag{15}$$

$$\Omega: -1 \leq u \leq 1$$

$$C(u) = t_1$$

$$G = \Big[(x_1, x_2):\ x_1(t) = 0 \text{ for all } t > 0 \text{ with } u(t) \in \Omega$$

where $x_1(t)$ is a solution of (15) with $x_1(0) = x_1,\ x_2(0) = x_2\Big]$

The problem statement is: find $u(x_1, x_2)$ so that $[x_1(t), x_2(t)]$ moves from any point $x_o \in R^2$ to the target G; moreover, in going from x_o to G, C(u) attains the smallest value with respect to $u \in \Delta$.

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*The condition that x(t) be bounded for all responses corresponding to u(t) in Δ is automatically satisfied by the linear system (9).

To apply the previous results, convexity must be established for G and the boundary of G must be found. A constructive procedure for doing this is outlined in ref. (9). The method described earlier in this report under the discussion of single-component control could also be used. Since $x_1(t)$ is to be 0 for $t > 0$ then $\dot{x}_1(t) = 0$ for $t > 0$. This implies that $u = -x_2$ for $t > 0$ with $|u| \leq 1$. Thus $\dot{x}_2 = -2x_2$ and $|x_2| \leq 1$ for all $t > 0$. Consider the points x_{2o} for which this will be true for $t > 0$. Because $x_2(t) = x_{2o}\, e^{-2t}$, any x_{2o} for which $|x_{2o}| \leq 1$ is satisfactory. Therefore, G is just the line segment $x_1 = 0$, $|x_2| \leq 1$. The target G is certainly compact, convex, and nonempty. It has also been demonstrated that $G \cap [K(t, x_o) - \partial K(t, x_o)] \neq \phi$ since for any point of G it is possible to pick $u = -x_2$ for $t > 0$, which is not an extremal control and must lead to a point interior to $K(t, x_o)$ for $t > 0$.

The previously outlined procedure will be used to solve for the optimum control u as $u(x_1, x_2)$. Consider

$$H = \eta_1 x_2 + (-\eta_2) x_2 + (\eta_1 + \eta_2) u,$$

where

$$\dot{\eta}_1 = -\frac{\partial H}{\partial x_1} = 0,$$

$$\dot{\eta}_2 = -\frac{\partial H}{\partial x_2} = -\eta_1 + \eta_2$$

Carrying out the condition of the maximum [equation (14)] yields

$$u = \operatorname{sgn}(\eta_1 + \eta_2)$$

from which the control is uniquely defined a. e., since $Bw = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $ABw = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are non-collinear. Substituting for u, the following equation systems are obtained with $t = -t$:

$$-\dot{x}_1 = x_2 + \operatorname{sgn}(\eta_1 + \eta_2)$$

$$-\dot{x}_2 = x_2 + \operatorname{sgn}(\eta_1 + \eta_2)$$

$$-\dot{\eta}_1 = 0$$

$$-\dot{\eta}_2 = -\eta_1 + \eta_2$$

Let $\eta_{1o} = \cos\theta$, $\eta_{2o} = \sin\theta$ with $0 \leq \theta \leq 2\pi$. Therefore, $\eta_1(t, \theta) =$ constant $= \eta_{1o} = \cos\theta$ and

$$\eta_2(t, \theta), = (\eta_{2o} - \eta_{1o})\, e^{-t} + \eta_{1o} = (\sin\theta - \cos\theta)\, e^{-t} + \cos\theta,$$

$$0 < \theta \leq 2\pi.$$

Let $v(t, \theta) = \eta_1(t, \theta) + \eta_2(t, \theta)$ and $u(t, \theta) = \operatorname{sgn} v(t, \theta)$. Consider $v(t, \theta)$ and the point $(0, 1)$ of G. Any line through the point $(0, 1)$ is a supporting line to G. For this point, allowable values of θ to satisfy the transversality condition are $\pi \leq \theta \leq 2\pi$. Calculate $u(t, \theta)$ along solution curves $[x_1(t), x_2(t)]$ starting at $(0, 1)$ for $\pi \leq \theta \leq 2\pi$.

$$u(t, \theta) = -1 \text{ for } \pi \leq \theta \leq 3/2\,\pi, \quad \text{all } t > 0$$

and

$$u(t, \theta) = +1 \text{ for } \theta = 2\pi$$

For each $3/2\pi \leq \theta \leq 2\pi$, the zeros of $v(t, \theta)$ occur at only one time, t_1, which attains all values between 0 and ∞ for $3/2\pi \leq \theta \leq 2\pi$; $u(t, \theta)$ is -1 for t in $(0, t_1)$ and + 1 for t in (t_1, ∞). Therefore,

$$\Gamma = \left[x_1(t), x_2(t): \dot{x}_1 = -x_2 + 1, \dot{x}_2 = x_2 + 1 \text{ with } x_1(0) = 0, x_2(0) = 1, t > 0\right]$$

constitutes the zeros of $v(t, \theta)$ in the x_1, x_2 plane for $\pi \leq \theta \leq 2\pi$, all $t > 0$, for optimum solutions to the point $(0, 1)$. For the point $(0, -1)$ of G, Γ is reflected through the origin to find $\hat{\Gamma}$, the zeros of $v(t, \theta)$ in the x_1, x_2 plane for $0 \leq \theta \leq \pi$, $t > 0$.

Finally consider points of G with $|x_2| < 1$. Here θ can only take the values 0, π, 2π; but $u(t, \theta)$ has no zeros for $\theta = 0, \pi, 2\pi$, all $t > 0$. Therefore, the zeros of $v(t, \theta)$ in the x_1, x_2 plane are exactly the set $\Gamma \cup \hat{\Gamma}$. It is easy to see that Γ, $\hat{\Gamma}$ and the line segment $|x_2| \leq 1$, $x_1 = 0$ form a line which divides the x_1, x_2 plane into two parts. On one side of this line $\operatorname{sgn}(\eta_1 + \eta_2)$ is -1, on the other +1, but this gives u as $u(x_1, x_2)$. Figure 1 shows the switching boundary and typical optimum solutions for this example.

SYSTEMS REPRESENTED BY LINEAR RECURRENCE EQUATIONS

As an approximation to optimum control of certain processes described by differential equation models, it has been suggested that recurrence equation models be considered. The study of the control of these equations is of value in its own right owing to the connection with various physical phenomena such as biological and economic processes. The degree of approximation to differential equation models is not known and is perhaps most easily determined by experiment. Concern here is only with the development of procedures for constructing the control function after the problem has been suitably set and only for the linear recurrence equation.

Similar problems have been considered previously (refs.11, 12, 13) and have led to results for time-optimum control to the origin. When the target is not the origin (for example, when it is a convex set), new methods must be used. The results presented here permit the generation of optimum trajectories in the same manner as for processes described by differential equations as considered above.

Recurrence Equations

Consider the real linear recurrence equation :

$$x^i(r+1) = A^i_j(r)\, x^j(r) + B^i_k(r)\, u^k(r) \qquad (16)$$

where

$$k = 1, 2, \ldots m$$
$$i, j = 1, 2, \ldots n$$
$$r = 0, 1, 2 \ldots$$

[or in vector form $x(r+1) = A(r)\, x(r) + B(r)\, u(r)$, $r = 0, 1, 2 \ldots$.] Here r denotes the stage in the evolution of the process. It is easy to show that (16) encompasses real scalar equations as in the case of differential equations.

First, for the real scalar equation :

$$x(r+n) + a_1(r)\, x(r+n-1) + \ldots + a_n(r)\, x(r) = y(r);\ r = 0, 1, 2 \ldots,\ a_n(r) \neq 0$$

the solution $x(r)$, $r = 0, 1, 2$, is the first component of the vector solution of the system

$$x^1(r+1) = x^2(r)$$
$$x^2(r+1) = x^3(r)$$
$$\vdots$$
$$x^n(r+1) = x(r+n) = -a_1(r)\, x^n(r) - a_2(r)\, x^{n-1}(r) + \ldots - a_n(r) x^1(r) + y(r)$$

Second, for the real scalar equation :

$$x(r+n) + a_1(r)\, x(r+n-1) + \ldots + a_n(r)\, x(r) = b_1(r)\, y(r+n-1) + b_2(r)\, y(r+n-2) + \ldots + b_n(r)\, y(r)$$

the solution $x(r)$, $r = 0, 1, 2, \ldots$ with initial data $[x(i), x(i+1), \ldots (x(i+n)]$ and $a_n(r) \neq 0$ is the first component of the vector solution of the system

$$x^1(r+1) = x^2(r) + h_1(r)\, y(r)$$
$$x^2(r+1) = x^3(r) + h_2(r)\, y(r)$$
$$\vdots$$
$$x^n(r+1) = -a_1(r)\, x^n(r) - a_2(r)\, x^{n-1}(r) - \ldots - a_n(r)\, x^1(r) + h_n(r)\, y(r)$$

$r = 0, 1, 2\ldots$. Here the $h_i(r)$ are found by elimination from the previous equations. This procedure is illustrated by considering a second-order example. Consider the real second-order scalar equation:

$$x(r+2) + a_1(r)\, x\,(r+1) + a_2(r)\, x(r) = b_1(r)\, y(r+1) + b_2(r)\, y(r)$$

Let

$$x(r) = x^1(r) + h_o(r)\, y(r) \tag{17}$$

and

$$x^1(r+1) = x^2(r) + h_1(r)\, y(r) \tag{18}$$

$$x^2(r+1) = -a_1(r)\, x^2(r) - a_2(r)\, x^1(r) + h_2(r)\, y(r) \tag{19}$$

From Equation (17)

$$x^1(r+1) = x(r+1) - h_o(r+1)\, y(r+1) = x^2(r) + h_1(r)\, y(r)$$

therefore,

$$x^2(r) = x(r+1) - h_o(r+1)\, y(r+1) - h_1(r)\, y(r)$$

Iterating one step,

$$x^2(r+1) = x(r+2) - h_o(r+2)\, y(r+2) - h_1(r+1)\, y(r+1) = -\, a_1(r)\, x^2(r) - a_2(r)\, x^1(r) + h_2(r)\, y(r)$$

Thus,

$$x(r+1) = x^2(r) + h_o(r+1)\, y(r+1) + h_1(r)\, y(r) \tag{20}$$

and

$$x(r+2) = h_o(r+2)\, y(r+2) + h_1(r+1)\, y(r+1) - a_1(r)\, x^2(r) - a_2(r)\, x^1(r) + h_2(r)\, y(r) \tag{21}$$

Multiplying equation (17) by $a_2(r)$, and equation (20) by $a_1(r)$ and then adding these two to equation (21),

$$\begin{aligned} x(r+2) + a_1(r)\, x(r+1) + a_2(r)\, x(r) &= h_o(r+2)\, y(r+2) + h_1(r+1)\, y(r+1) - a_1(r)\, x^2(r) \\ &- a_2(r)\, x^1(r) + h_2(r)\, y(r) + a_1(r)\, x^2(r) + a_1(r)\, h_o(r+1)\, y(r+1) + a_1(r)\, h_1(r) y(r) \\ &+ a_2(r)\, x^1(r) + a_2(r)\, h_o(r)\, y(r) = b_1(r)\, y(r+1) + b_2(r)\, y(r) \end{aligned}$$

Since y(r), y(r+1), and y(r+2) are linearly independent (by assumption), the coefficients of each of these terms must equal zero; i. e.,

$$h_o(r+2) = 0$$

$$h_1(r+1) + a_1(r)\, h_o(r+1) = b_1(r)$$

and

$$h_2(r) + a_1(r)\, h_1(r) + a_2(r)\, h_o(r) = b_2(r)$$

Therefore,

$$h_1(r) = b_1(r-1)$$

and

$$h_2(r) = b_2(r) - a_1(r)\, b_1(r-1)$$

The equation system is

$$x^1(r+1) = x^2(r) + b_1(r-1)\, y(r)$$

$$x^2(r+1) = -a_1(r)\, x^2(r) - a_2(r)\, x^1(r) + \left[b_2(r) - a_1(r)\, b_1(r-1)\right] y(r)$$

with initial data $x^1(0) = x(0)$, and $x^2(0) = x(1) - h_1(0)\, y(0)$

Hereafter, solutions will be considered only for the system of equation (16) since the above shows (16) is sufficiently general to encompass real scalar equations. Some general results will now be presented concerning the nature of solutions of (16). In particular, a variation-of-parameters formula is given in theorem 6.

Consider the homogeneous equation

$$x(r+1) = A(r)\, x(r); \quad r = 0, 1, 2...$$

The set of all solutions forms an n-dimensional real vector space. Writing $(x) = x(0), x(1), x(2)...$ for the solution with initial data x(0),

$$a(x) + b(\bar{x}) = a\, x(0) + b\, \bar{x}(0),\ ax(1) + b\bar{x}(1), .., a\, x(n) + b\bar{x}(n), ..$$

The solution $a(x) + b(\bar{x}) = 0$ if and only if $ax(0) + b\bar{x}(0) = 0$. Therefore, it is necessary to find a basis only for the initial vector x(0). Note also that an nxn real matrix W(r) is a solution of $W(r+1) = A(r)W(r)$, $r = 0, 1, 2, ...$ if and only if each column of W(r) is a solution of the vector equation $x(r+1) = A(r)\, x(r)$. A matrix solution W(r) with W(0) nonsingular is called a fundamental matrix solution.

2

Theorem 3. - Consider the homogeneous real linear equation

$$x(r+1) = A(r)\, x(r) \text{ in } R^n$$

$r = 0, 1, 2 \ldots$. A set of n solutions is a basis for the space of solutions if, and only if, they form the columns of a fundamental solution matrix.

Proof: Let

$$W(r) = \left[x^{(1)}(r),\ x^{(2)}(r), \ldots x^{(n)}(r)\right]$$

Then

$$x^{(1)}(r),\ x^{(2)}(r), \ldots x^{(n)}(r)$$

form a basis of solutions if, and only if, $\left[x^{(1)}(0),\ x^{(2)}(0) \ldots x^{(n)}(0)\right]$ are linearly independent; that is, $W(0)$ is nonsingular.

Remark: If $W(r)$ is one fundamental solution matrix, then all other fundamental solution matrices are just $W(r)C$, where C is a constant nxn nonsingular matrix.

Proof: First show that $W(r)C$ is a solution; i. e., it satisfies the recurrence equation.

$$W(r+1)C = A(r)\, W(r)C; \quad r = 0, 1, 2 \ldots$$

Now let $Y(r)$ be a fundamental solution matrix. Then

$$|Y(0)| \neq 0$$

and

$$Y(0)Y^{-1}(0) = I$$

Choose

$$C = W^{-1}(0)\, Y(0)$$

then

$$Y(r) = W(r)C = W(r)\left[W^{-1}(0)\, Y(0)\right]$$

and

$$Y(0) = W(0)\left[W^{-1}(0)\, Y(0)\right] = Y(0) \text{ Q. E. D.}$$

Remark: $W(0) = I$, $W(1) = A(0)$, $W(2) = A(1)\,A(0)$, $W(3) = A(2)\,A(1)\,A(0) \ldots$, $W(r+1) = A(r)\,A(r-1) \ldots A(1)\,A(0), \ldots$ is a fundamental solution matrix.

If A is a constant matrix, a basis of solutions can easily be found. Consider $x(r+1) = Ax(r)$ with A a constant nxn real matrix. Then from the above remark, $W(r) = (A)^r$ is a fundamental solution matrix. If P is any nonsingular matrix, then $y(r) = (PAP^{-1})^r$ is a fundamental solution matrix of $z(r+1) = PAP^{-1}z(r)$. If A is similar to a diagonal matrix (say A is symmetrical) there exists a P such that

$$PAP^{-1} = D = \text{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Each column of

$$P^{-1}(D)^r = P^{-1}\,\text{diag}(\lambda_1^{\ r}, \lambda_2^{\ r}, \ldots \lambda_n^{\ r})$$

is a solution of $x(r+1) = Ax(r)$. The most general solution is an arbitrary linear combination of the columns:

$$z^{(1)}(r) = \begin{bmatrix} \lambda_1^{\ r} \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad z^{(2)}(r) = \begin{bmatrix} 0 \\ \lambda_2^{\ r} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad \ldots \qquad z^{(n)}r = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \lambda_n^{\ r} \end{bmatrix}$$

that is,

$$z(r) = c_1 z^{(1)}(r) + c_2 z^{(2)}(r) + \ldots + c_n z^{(n)}(r)$$

Another method of solving for the constant case is to assume a solution and see if it is possible. Assume that $\lambda^r v$ is a solution where v is a non-zero constant vector. Then substituting in the recurrence equation, λ must be such that

$$\lambda^{r+1} v = A\lambda^r v \quad \text{or} \quad \left| A - \lambda I \right| = 0$$

Any root λ_i of this equation leads to a solution $x(r) = \lambda_i^{\ r} v_i$. If these roots are distinct, the most general solution is

$$x(r) = c_1\lambda_1^{\ r} + c_2\lambda_2^{\ r} + \ldots + c_n\lambda_n^{\ r}$$

2

A result concerning stability for the constant coefficient linear recurrence equation is given in theorem 4.

Theorem 4. - Consider the real autonomous recurrence equation

$$x(r+1) = Ax(r) \text{ in } R^n$$

If every eigenvalue of A has a modulus less than 1, A is called a strict contraction and the origin (0) of R^n is asymptotically stable (i. e. , 0 is the unique critical point and every solution tends to this point as r increases).

For the nonhomogeneous equation, theorem 5 applies.

Theorem 5. - Consider the real recurrence equation

$$x(r+1) = A(r)\, x(r) + B(r)\, u(r)$$

$r = 0, 1, 2...$ in R^n. Let $x^{(P)}(r)$ be a solution. Let $x^{(1)}(r)$, $x^{(2)}(r), \ldots x^{(n)}(r)$ be a basis of solution for the corresponding homogeneous equation.

$$x(r+1) = A(r)\, x(r)$$

Then the most general solution is

$$x(r) = c_1 x^{(1)}(r) + c_2 x^{(2)}(r) + \ldots + c_n x^{(n)}(r) + x^{(P)}(r)$$

$r = 0, 1, 2...$.

Proof: Let $x^{(P)}(r)$ and $\hat{x}^{(P)}(r)$ be solutions of

$$x(r+1) = A(r)\, x(r) + B(r)\, u(r)$$

then $x_r^{(P)} - \hat{x}^{(P)}(R)$ is a solution of the homogeneous equation, that is,

$$x^{(P)}(r) - \hat{x}^{(P)}(r) = c_1 x^{(1)}(r) + c_2 x^{(2)}(r) + \ldots + c_n x^{(n)}(r)$$

Theorem 6. - Consider

$$x(r+1) = A(r)\, x\, (r) + B(r)\, u(r)$$

$r = 0, 1, 2...$ in R^n. Let c be a given real constant n-vector. Then the unique solution with $x(0) = c$ is

$$x(r) = W(r)c + W(r) \sum_{j=1}^{r} W^{-1}(j)\, B(j-1)\, u(j-1)$$

2

Note that $W(r) = A(r-1)\,A(r-2),\ldots\,A(0)$ and

$$W(r)\,W^{-1}(j) = A(r-1)\cdot A(r-2)\ldots A(0)\cdot A^{-1}(0)\ldots A^{-1}(j-1) = A(r-1)\cdot A(r-2)\ldots A(j)$$

Proof: This variation-of-parameters formula is proved by induction. For $r=0$ $x(0) = c$, and $r=1$; $x(1) = A(0)c + B(0)\,u(0)$. If this is assumed true for r, then

$$W(r+1)c + W(r+1)\sum_{j=1}^{r+1} W^{-1}(j)\,B(j-1)\,u(j-1) = A(r)\,W(r)c +$$

$$A(r)\,W(r)\sum_{j=1}^{r} W^{-1}(j)\,B(j-1)\,u(j-1) + W(r+1)\,W^{-1}(r+1)\,B(r)\,u(r)$$

$$= A(r)\,x(r) + B(r)\,u(r) \quad \text{Q. E. D.}$$

The Control Problem

Consider

a. The real linear recurrence equation $x(r+1) = A(r)\,x(r) + B(r)\,u(r)$ $[A(r),\ B(r)$ bounded$]$ $r = 0, 1, 2\ldots$

b. The restraint on the control Ω:
$|u^k(r)| \leq 1$, $k = 1, 2\ldots m$, $r = 0, 1, 2\ldots$

c. The initial data $x_o = x(0) = x^1(0),\ x^2(0),\ldots x^n(0)$.

d. The target set G as a convex, compact, nonempty subset of R^n.

e. The cost functional of control $C(u) = r$.

The problem of optimum control is to select the sequence of numbers $u^k(r)$; $r = 0, 1, 2\ldots$ from Ω at each stage r so that $x(r)$ with $x(0) = x_o$ is a point of G for minimum r. Later an equivalent definition of optimum control is given which will provide a method for constructing the optimum control for this problem.

Let $W(r)$ be a fundamental matrix for the homogeneous equation $x(r+1) = A(r)\,x(r)$, with $W(0) = I$.

Definition: The set of points

$$K(r,x_o) = \left[x : x = W(r)\,x_o + W(r)\sum_{j=1}^{r} W^{-1}(j)\,B(j-1)\,u(j-1);\ u(j)\text{ allowable}\right]$$

is called the set of attainability from x_o in r stages, $r = 0, 1, 2 \ldots$.

The set $K(r,x_o)$ has the following properties.

Lemma 6. - Consider the set $K(r,x_o)$, $r = 0, 1, 2 \ldots$. For $u(j)$ restricted to Ω; $K(r,x_o)$ is a compact, convex, nonempty subset of R^n.

Proof: Obviously $K(r,x_o)$ is a nonempty subset of R^n. To show that it is convex, let $[u(0),\ u(1), \ldots u(r-1)]$ and $[\hat{u}(0),\ \hat{u}(1), \ldots \hat{u}(r-1)]$ be two controls with corresponding responses $[x(1),\ x(2), \ldots x(r)]$ and $[\hat{x}(1),\ \hat{x}(2), \ldots \hat{x}(r)]$. Consider the points on the line between $x(r)$ and $\hat{x}(r)$; i.e.

$$\widetilde{x}(r) = \lambda x(r) + (1-\lambda)\,\hat{x}(r),\ \ 0 \le \lambda \le 1$$

Each of these points belongs to $K(r,x_o)$ by use of the control sequence $[\lambda u(0) + (1-\lambda)\hat{u}(0),\ \lambda u(1) + (1-\lambda)\hat{u}(1), \ldots, \lambda u(r-1) + (1-\lambda)\,\hat{u}(r-1)]$. Therefore, $K(r,x_o)$ is convex.

To show that $K(r, x_o)$ is compact, consider the compact set

$$\Lambda = \Omega \times \Omega \times \Omega \ldots \times \Omega \text{ in } R^{rm}$$

Each point of Λ defines a control sequence $[u(0),\ u(1), \ldots u(r-1)] = u$. Consider the linear function f on Λ to R^n

$$f = W(r)\sum_{j=1}^{r} W^{-1}(j)\,B(j-1)\,u(j-1) + W(r)x_o$$

Then $K(r,x_o) = f(\Lambda) \subset R^n$ is compact by a well-known theorem about continuous functions (see, for example, ref. 14, page 141).

Let

$$\omega(\eta_o) = W(r)\,x_o + W(r)\sum_{j=1}^{r} W^{-1}(j)\,B(j-1)\,\operatorname{sgn}\left\{\left[W^{-1}(j)\,B(j-1)\right]'\eta_o\right\}$$

(' indicates transpose).

<u>Lemma 7.</u> - Consider $\eta_r = \left[W^{-1}(r)\right]'\eta_o$ any non-zero vector in R^n

$$\eta_r \cdot \omega(\eta_o) \geq \eta_r \cdot \zeta \text{ for all } \zeta \epsilon K(r, x_o)$$

Proof:

$$\eta_r \cdot \omega(\eta_o) - \eta_r \cdot \zeta = \eta_o \cdot x_o + \sum_{j=1}^{r} \left[W^{-1}(j)\, B(j-1)\right]'\eta_o \cdot \operatorname{sgn}\left\{\left[W^{-1}(j)\, B(j-1)\right]'\eta_o\right\} -$$

$$\eta_o \cdot x_o + \sum_{j=1}^{r} \left[W^{-1}(j)\, B(j-1)\right]'\eta_o \cdot u(j-1) = \sum_{j=1}^{r} \left|\left[W^{-1}(j)\, B(j-1)\right]'\eta_o\right| -$$

$$\sum_{j=1}^{r} \left[W^{-1}(j)\, B(j-1)\right]'\eta_o \cdot u(j-1)$$

But term by term,

$$\left|\left[W^{-1}(j)\, B(j-1)\right]'\eta_o\right| \geq \left[W^{-1}(j)\, B(j-1)\right]'\eta_o \cdot u(j-1)$$

for $\left|u(j)\right| \leq 1$. Therefore

$$\eta_r \cdot \omega(\eta_o) \geq \eta_r \cdot \zeta \text{ , all } \zeta \epsilon K(r, x_o)$$

Remark: If

$$\left[W^{-1}(r)\right]'\eta_o^{\,1} \cdot \omega(\eta_o^{\,1}) = \left[W^{-1}(r)\right]'\eta_o^{\,1} \cdot \omega(\eta_o^{\,2})$$

for any two unit n-vectors $\eta_o^{\,1}$ and $\eta_o^{\,2}$, then

$$\operatorname{sgn}\left\{\left[W^{-1}(j)\, B(j-1)\right]'\eta_o^{\,2}\right\} = \operatorname{sgn}\left\{\left[W^{-1}(j)\, B(j-1)\right]'\eta_o^{\,2}\right\}$$

$j = 1, 2 \ldots r$, except possibly when

$$\left[W^{-1}(j)\, B(j-1)\right]'\eta_o^{\,1} = 0$$

This shows that to go the farthest in any direction in R^n, an extremal control must be used; that is, equal to sgn$[\ ]$ whenever $[\ ] \neq 0$, and arbitrary in Ω at other stages.

Definition: Equation (16) is called irrational of degree N if every n collection of the vectors $\left[W^{-1}(j)\ B(j-1)\right]_k$ $k = 1, 2 \ldots m$; $j = 0, 1, 2 \ldots N$, has span equal to R^n.

Theorem 7. - If equation (16) is irrational, there is a one-to-one correspondence, for $0 \leq r \leq N$, between points of $\partial K(r, x_o)$ and the extremal controls:

$$\begin{aligned} u(j-1) &= 1 \text{ if } \left[W^{-1}(j)B(j-1)\right]' \eta_o > 0 \\ &= a \text{ if } \left[W^{-1}(j)B(j-1)\right]' \eta_o = 0 \\ &= -1 \text{ if } \left[W^{-1}(j)B(j-1)\right]' \eta_o < 0 \end{aligned}$$

for $j = 1, 2 \ldots r$, $\|\eta_o\| = 1$, and $|a| \leq 1$

Proof: That each extremal control leads to a point in $\partial K(r, x_o)$ is proved by extending lemma 7. Suppose

$$u = \left[u(0),\ u(1),\ \ldots\ u(r-1)\right],$$

depending on η_o, was an extremal control which led to an interior point $\omega(\eta_o)$ of $K(r, x_o)$. Consider

$$\eta_r = \left[W^{-1}(r)\right]'\eta_o$$

By lemma 7, $\eta_r \cdot \omega(\eta_o) \geq \eta_r \cdot \zeta$ for all $\zeta \in K(r, x_o)$. Therefore, $\omega(\eta_o)$ cannot be an interior point and must then be in $\partial K(r, x_o)$.

To show that the control u which leads to $x(r) \in \partial K(r, x_o)$ is necessarily extremal, let π be any support plane to the convex set $K(r, x_o)$ at $x(r) \in \partial K(r, x_o)$. Choose η_r to be a non-zero vector normal to π and directed into the halfspace not containing $K(r, x_o)$. Compute.

$$\eta_r \cdot x_r = \eta_r \cdot \left\{ W(r)x_o + W(r) \sum_{j=1}^{r} \left[W^{-1}(j)B(j-1)\right] u(j-1) \right\} = \eta_r \cdot \omega(\eta_o)$$

Therefore,

$$\eta_o \cdot x_o + \sum_{j=1}^{r} \left[W^{-1}(j)B(j-1)\right]' \eta_o \cdot \operatorname{sgn}\left\{\left[W^{-1}(j)B(j-1)\right]' \eta_o\right\} - \eta_o \cdot x_o +$$

$$\sum_{j=1}^{r} \left[W^{-1}(j)B(j-1)\right]' \eta_o \cdot u(j-1) = \sum_{j=1}^{r} \left|\left[W^{-1}(j)B(j-1)\right]' \eta_o\right| -$$

$$\sum_{j=1}^{r} \left[W^{-1}(j)B(j-1)\right]' \eta_o \cdot u(j-1)$$

Since u(j-1) is in each component bounded by one, it is necessary that

$$u^k(j-1) = \operatorname{sgn}\left\{\left[W^{-1}(j)B(j-1)\right]' \eta_o\right\}^k$$

whenever

$$\left\{\left[W^{-1}(j)B(j-1)\right] \eta_o\right\}^k \neq 0$$

$$k = 1, 2 \ldots n$$

in order to obtain zero for the above difference. Therefore, all controls which lead to $x(r) \in \partial K(r, x_o)$ are necessarily extremal.

Because each extremal control leads to a unique response (theorem 6), it leads to only one point of the boundary of $K(r, x_o)$.

It must now be established that for each point $x(r) \in \partial K(r, x_o)$ there is only one extremal control. Let π, as above, be any support plane to $K(r, x_o)$ at $x(r) \in \partial K(r, x_o)$ and

$$\eta_r = \left[W^{-1}(r)\right]' \eta_o$$

be a non-zero normal vector to π directed to the halfspace not containing $K(r, x_o)$. Consider

$$x(r) = W(r)\, x_o + W(r) \sum_{v, j:\ \left\{\left[W^{-1}(j)B(j-1)\right]' \eta_o\right\}^v \neq 0} \left[W^{-1}(j)B(j-1)\right]_v \operatorname{sgn}\left\{\left[W^{-1}(j)B(j-1)\right]' \eta_o\right\}^v +$$

$$W(r) \sum_{v, j:\ \{\ \}^v = o} \left[W^{-1}(j)B(j-1)\right]_v a^v(j-1), \quad \left|a^v(j-1)\right| \leq 1$$

Since $x(r) \in \partial K(r, x_o)$ such representation for $x(r)$ is always possible. Suppose

$$\eta_r \cdot \omega(\eta_o) = \eta_r \cdot \omega(\hat{\eta}_o),$$

$$\omega(\eta_o) = \omega(\hat{\eta}_o)$$

Then from the remark following lemma 7,

$$\operatorname{sgn}\left\{\left[W^{-1}(j)B(j-1)\right]'\,\hat{\eta}_o\right\}^v = \operatorname{sgn}\left\{\left[W^{-1}(j)B(j-1)\right]'\,\eta_o\right\}^v$$

except when

$$\left\{\left[W^{-1}(j)B(j-1)\right]'\,\eta_o\right\}^v = 0$$

where

$$\eta_r = \left[W^{-1}(r)\right]'\,\eta_o$$

and

$$\hat{\eta}_r = \left[W^{-1}(r)\right]'\,\hat{\eta}_o$$

are both exterior normals to the support planes $\hat{\pi}$ and π at

$$\omega(\eta_o) = \omega(\hat{\eta}_o) = x(r) \in \partial K(r, x_o).$$

It then only remains to establish that the equation

$$W^{-1}(r)x(r)-x_o - \sum_{v,j:\{\}\neq o}\left[W^{-1}(j)B(j-1)\right]_v \operatorname{sgn}\left\{\left[W^{-1}(j)B(j-1)\right]'\,\eta_o\right\}^v =$$

$$\sum_{v,j:\{\}=o}\left[W^{-1}(j)B(j-1)\right]_v a^v(j-1), \quad \left|a^v(j-1)\right| \leq 1$$

$j = 1, 2. \ldots r$

has a unique solution in terms of the $a^v(j-1)$. Because the system is irrational, it is a system of n linear equations in $\ell \leq n-1$ unknowns. Since a solution exists, it is only necessary to show that it is unique. This is obviously true since any n collection of vectors $[W^{-1}(j)B(j-1)]_k$ $k = 1, 2, \ldots m$, $j = 1, 2, \ldots n$ forms a basis for R^n and therefore any $\ell \leq n-1$ collection of these same vectors form a basis for an ℓ dimensional linear subspace of R^n.

To construct optimum control based on the previous theorem, it is advantageous to modify the original definition of optimum control. To do this, a dimension is added to the target set G:

$$G(a, r) = G \cup \left[x(r-1): x(r) = A(r-1)x(r-1) + B(r-1)u(r-1);\ x(r) \in \partial G,\ 0 < \left|u(r-1)\right| < a\right]$$

$$0 \leq a \leq 1$$

Definition: The control $\left[u(0),\ u(1),\ \ldots\ u(r-2)\right]$ which steers x_o to $G(a, r)$ is considered optimum in case $C(u) = r-1+a$ is a minimum.

Uniqueness of the optimum control is obtained in the following case.

Theorem 8. -Consider equation (16) irrational. Let

$$u = u(0),\ u(1),\ \ldots\ u(r-2)$$

and

$$\hat{u} = \hat{u}(0),\ \hat{u}(1),\ \ldots\ \hat{u}(r-2)$$

be two controls which steer $x(r)$ from $x(0)$ to $G(a, r)$ optimally. If ∂G contains no line segments parallel to any of the vectors

$$\left[W^{-1}(j)B(j-1)\right]_k,\quad k = 1,\ 2, \ldots\ m,\quad j = 1,\ 2,\ \ldots\ r,$$

then $u = \hat{u}$; that is $u(0) = \hat{u}(0)$, $u(1) = \hat{u}(1)$, ...

Proof: Let r^* be the first $r > 0$ for which $K(r-1, x_o) \cap G(1, r) \neq \phi$. Clearly this is the optimum r. By construction,

$$K(r^* - 1,\ x_o) \cap G = \phi$$

Because of the natural ordering $G \subset G(a, r^*) \subset G(a_2, r^*)$ for $0 \leq a \leq a_2 \leq 1$, the minimum of $C(u)$ occurs with the smallest $0 \leq a \leq 1$ for which $K(r^*-1, x_o) \cap G(a, r^*) \neq \phi$. Through each point of $G(1, r^*) - G$ passes at least one of the surfaces $\partial G(a, r^*)$, $0 \leq a \leq 1$. Therefore the optimum $C(u)$ can be found by finding the smallest $0 \leq a \leq 1$ for which $\partial K(r^* -1,\ x_o) \cap G(a, r^*) \neq \phi$. Call this optimum a^*. This first intersection contains only one point $x(r^*-1)$ because ∂G and therefore $\partial G(a^*, r^*)$ contains no line segments parallel to line segments of $\partial K(r^*-1,\ x_o)$. By theorem 7 for each point $x(r-1) \in \partial K(r-1, x_o)$ there is only one control for irrational systems. Therefore, $u = \hat{u}$.

The following theorem provides a sufficient condition for optimum control.

Theorem 9. - Consider irrational equation (16) with the boundary of the convex set G containing no line segments parallel to any of the vectors

$$\left[W^{-1}(j)B(j-1)\right]_k,\quad k = 1, 2, \ldots m;\ j = 1, 2, \ldots r^*.$$

Let the control

$$u = \left[u(0),\ u(1),\ u(2), \ldots\ u(r-2)\right]$$

which steers $x(r)$ from x_o to $\partial G(a, r)$, $0 \leq a \leq 1$ be such that

1. u is an extremal control
2. $\eta_{r-1} = [W^{-1}(r-1)]'\eta_o, \; \|\eta_o\| = 1$, is normal to a supporting hyperplane of $G(a, r)$ at $x(r-1) \epsilon \partial G(a, r)$ and directed into the halfspace containing G, $[x(r-1) \epsilon \partial K(r, x_o)]$
3. $G \cap [K(r, x_o) - \partial K(r, x_o)] \neq \phi$ all $r > r^*$; then u is the unique optimum control.

Proof: Consider all $0 \leq r$ and $0 \leq a \leq 1$ for which $K(r-1, x_o)$ and $G(a, r)$ have a common support plane with $K(r-1, x_o)$ on one side and $G(a, r)$ on the other. Since $G \subset G(a, r)$, $0 \leq a \leq 1$ all $r > 0$ and $G \cap [K(r, x_o) - \partial K(r, x_o)] \neq \phi$, $r > r^*$, there exists no such support plane for $r > r^*$ because η_{r-1} is an exterior normal to $K(r-1, x_o)$ by lemma 7. Therefore $r \leq r^*$. But r^* is the first $r > 0$ for which $K(r-1, x_o) \cap G(r) \neq \phi$. Hence, only for $r = r^*$ does such a support plane exist. Let π be a common support plane for $K(r^*-1, x_o)$ and $G(a, r^*)$, $0 \leq a \leq 1$. Since $G(a^*, r^*) \subset G(a, r^*)$ for $a > a^*$, π is a support plane for $G(a^*, r^*)$. Also ∂G, and hence $\partial G(a^*, r)$, contains no line segments parallel to line segments of $\partial K(r^*-1, x_o)$; therefore,

$$K(r^*-1, x_o) \cap G(a^*, r^*) = x(r^*-1) \epsilon \partial K(r^*-1, x_o).$$

Choose

$$\eta_{r^*-1} = \left[W^{-1}(r^*-1) \right]' \eta_o, \; \left\| \eta_o \right\| = 1$$

normal to π and directed into the halfspace containing $G(a^*, r^*)$. By lemma 7 and theorem 7 corresponding to any such η_o there is only one extremal control which leads to $x(r^*-1) \epsilon \partial K(r^*-1, x_o)$. Therefore the extremal control $u = u_{\eta_o}$ is optimum and, by theorem 8, unique.

Remarks: $G \cap [K(r, x_o) - \partial K(r, x_o)] \neq \phi$ for $r > 0$ if there exists $\left| u^k(r) \right| \leq 1$, $k = 1, 2, \ldots m$ such that $x_{r+1} \epsilon G$ for each $x_r \epsilon \partial G$, $r = 0, 1, 2 \ldots$.

Theorem 9 can be used in constructing the optimum feedback function $u = u(x)$ when the matrices A and B are constant. In this case the method indicated for linear differential systems can be used to run the system backward to generate optimum trajectories.

CONCLUSIONS

The time-optimal control problem considered for linear differential equation systems is sufficiently general so that it may represent either the single-, multiple- or all-component time-optimal regulator problems. For this problem Pontryagin's maximum principle and associated transversality condition form conditions which an optimal control must satisfy. With the addition of a condition concerning the nature of the target set, it is shown that these necessary conditions are also sufficient. A method of synthesis is presented, and a second-order example is treated by this method.

The analogous problem is considered for systems represented by linear recurrence equations, and analogous results are obtained.

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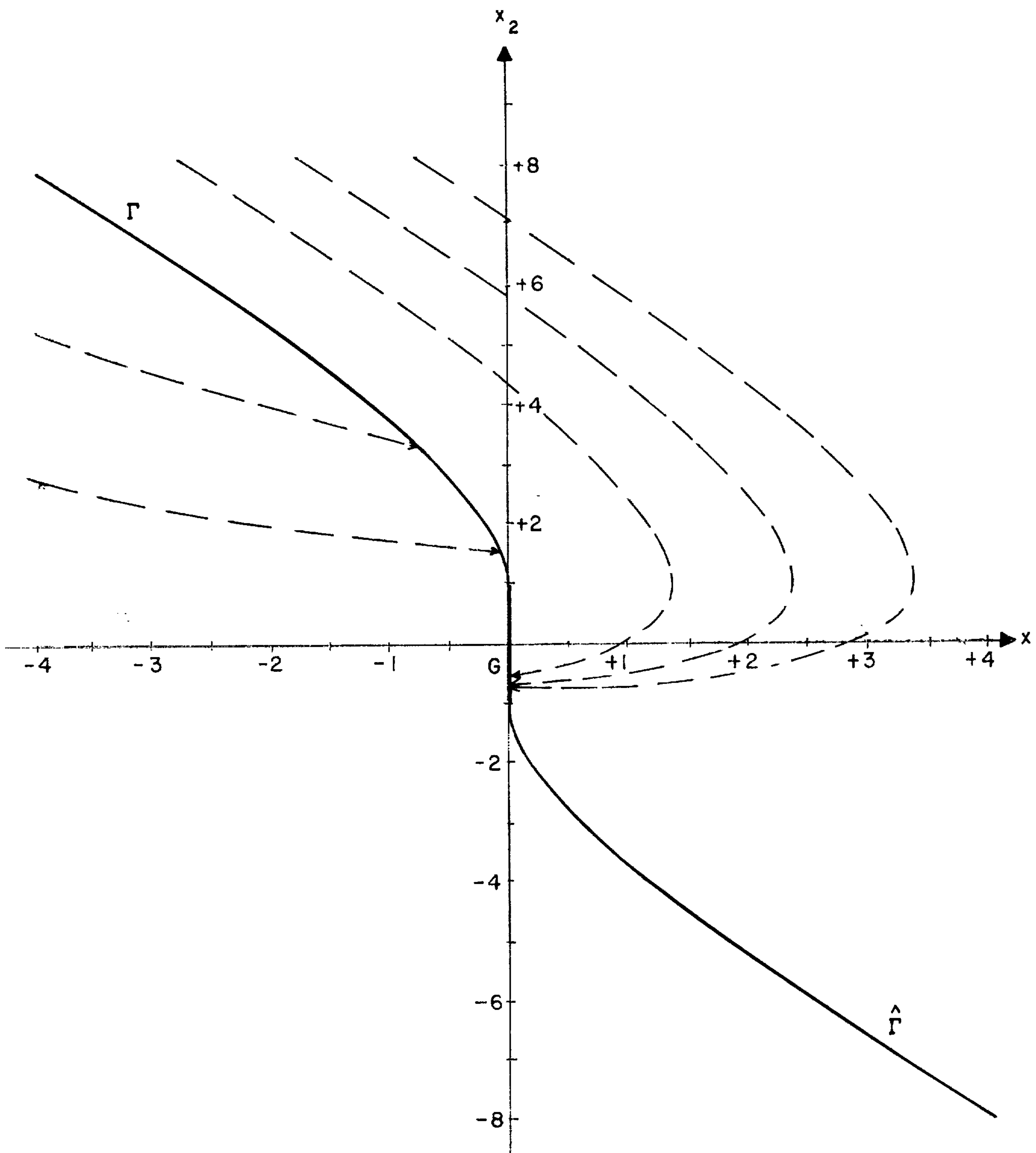

Figure 1. Phase-Plane Portrait of a Time-Optimal Controller